

Symmetry of Lepton Mixing

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Abstract

Neutrino mixing is studied from a symmetry perspective, both bottom-up and top-down. In the bottom-up approach, we start from the tri-bimaximal mixing, or one of its three partial patterns, and construct a list of horizontal symmetry groups capable of reproducing the mixing without adjustment of parameters. This list, labeled by an integer $n \geq 3$, is explicitly calculated for $n = 3$. In the top-down approach, we start from any finite group possessing a three-dimensional irreducible representation and an order-2 element, give a recipe to determine what mixing pattern it contains, and how to construct a dynamical model to reveal a particular mixing. Finally, we point out that if quark mixing is controlled by symmetry in this way, then there is an exciting possibility to determine most of the CKM mixing parameters by symmetry alone.

I. INTRODUCTION

Neutrino mixing can be described by the tri-bimaximal PMNS matrix [1]

$$U = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ -1 & \sqrt{2} & -\sqrt{3} \end{pmatrix}. \quad (1)$$

It predicts a zero reactor angle θ_{13} , a maximal atmospheric angle with $\sin^2 \theta_{23} = 0.50$, and a solar angle with $\sin^2 \theta_{12} = 0.333$. These predictions agree with the measured values $\sin^2 \theta_{13} = 0.9_{-0.9}^{+2.3} \times 10^{-2}$, $\sin^2 \theta_{23} = 0.44(1_{-0.22}^{+0.41})$, and $\sin^2 \theta_{12} = 0.314(1_{-0.15}^{+0.18})$ deduced from a global fit of the experimental data [2] to better than one standard deviation.

There are many attempts to produce an acceptable mixing from a horizontal group [3–7]. A successful model should yield the mixing pattern (1) automatically (or something equal to it within experimental errors) without having to tune any parameter. Parameters present should only be used to fit the neutrino mass gaps but not the mixing pattern. A partially successful model would be one that reproduces some but not all the features of (1), in which case the parameters have to be used to tune the mass gaps as well as the remaining features of mixing. With these criteria in mind, the degree of success of the existing models varies, but even when they are (partially) successful, it is not always clear what is responsible for the success because there are so many adjustable inputs. Is it the choice of a particular horizontal group, the choice of particular irreducible representations, and/or the choice of certain Higgs expectation patterns? Sometimes different choices can also lead to very similar results, and why is that so? There is also the important but difficult problem of understanding the significance of the mixing pattern (1). Why is it that way but not something else completely arbitrary?

In this work we attempt to answer some of these questions in a bottom-up approach, in which both (1) and three of its partial patterns are studied. The latter are bimaximal mixing without trimaximal mixing, trimaximal mixing without bimaximal mixing, and a third pattern specified by the first column of U in (1), just like bimaximal and trimaximal mixing are respectively specified by the third and the second columns. We shall label these patterns by an index α : the full pattern is $\alpha = 0$, and the partial pattern specified by column i of (1) is $\alpha = i$. For a partial pattern, one column of the mixing matrix is fixed, and the other two are arbitrary, subject only to unitarity considerations.

For a given α , the first question to ask is what finite horizontal groups \mathcal{G}^α can automatically give rise to that mixing pattern. A method to construct a list is discussed in the next section. The resulting group, labeled by an integer $n \geq 3$, will be denoted \mathcal{G}_n^α . Some (α, n) may yield no group, while others may give rise to more than one. The list may turn out to be finite or infinite. For $n = 3$, the list is

$$\mathcal{G}_3^0 = \mathcal{G}_3^1 = \{S_4, H(12, 3)\}, \quad \mathcal{G}_3^2 = \{A_4\}, \quad \mathcal{G}_3^3 = \{S_3, H(6, 3)\}, \quad (2)$$

where S_k is the symmetric (permutation) group of k objects, and A_k is the corresponding alternating (even permutation) group. The group $H(m, n)$ is defined by two generators F and G with the relations $G^2 = F^n = (FG)^m = E$ (identity matrix) [8]. In particular, it is known that [9] $H(2, 2) = Z_2 \times Z_2$, $H(4, 2) = D_4$, $H(2, 3) = S_3$, $H(3, 3) = A_4$, and $H(4, 3) = S_4$. The notations Z_m and D_m stand for the cyclic and the dihedral group respectively. The group $H(6, 3)$ has 54 elements, and the group $H(12, 3)$ has 216 elements. Clearly any group that contains a subgroup in this list will do as well. For example, $A_4 \in \mathcal{G}_3^2$ can guarantee a trimaximal mixing ($\alpha = 2$) but not a bimaximal mixing ($\alpha = 3$). However, since $S_3, A_4 \in S_4$, the group S_4 can produce not only the mixing pattern $\alpha = 1$, but also trimaximal and bimaximal mixing, and that is why $\mathcal{G}_3^1 = \mathcal{G}_3^0$.

The next question to ask is what \mathcal{G} -representations and Higgs expectation patterns are needed to yield the specific mixing pattern. This is discussed in Sec. 3, using an approach in which the right-handed and all heavy leptons have been integrated out, so only left-handed leptons and Higgs remain. With two exceptions which will be discussed in Sec. 3, the left-handed leptons are always assigned a three-dimensional irreducible representation, but it does not matter how many Higgs are present and what representations they belong to, as long as their expectation values are invariant under the appropriate residual symmetry operators to be discussed in Secs. 2 and 3. For dynamical models with the presence of right-handed and/or heavy leptons, the present formalism gives only the constraint placed on the effective model after these other leptons are integrated out.

As to the significance of having the tri-bimaximal mixing pattern rather than something arbitrary, we can offer the following observation which will be elaborated in Sec. 3. Any *finite* horizontal group that contains the requisite residual symmetries can only yield a small number of possible mixing patterns; the tri-bimaximal pattern (1) and its sub-patterns are among those possible for the groups in \mathcal{G}_n^α . In general, the smaller the horizontal group,

the more limited is the number of possible mixing patterns. Furthermore, the choice of horizontal groups is quite limited, as it must be a group containing an order-2 element to act as the neutrino residual symmetry, and a three-dimensional irreducible representation for the leptons to reside in. Thus, although symmetry alone cannot determine mixing, it can pick out a discrete number of possibilities to which (1) and its subpatterns belong.

That also means that a mixing pattern which deviates from (1) by a small amount in all three columns is expected to come either from an infinite horizontal group, or else the residual symmetries are softly but weakly broken.

It would be very exciting if quark mixing can be explained by symmetry in the same way, for the discreteness of the allowed mixings makes it conceivable to have most of the CKM parameters determined that way by symmetry alone. This point is further discussed in the concluding section.

II. RESIDUAL AND HORIZONTAL SYMMETRIES

Since symmetries normally refer to Hamiltonians, in this case mass matrices, it is useful to find out how to translate mixing patterns into mass-matrix symmetries. In the basis of a diagonal charged-lepton mass matrix (squared) $M_e M_e^\dagger$, the Majorana mass matrix M_ν for the active neutrinos can be diagonalized by the mixing matrix U to produce $U^T M_\nu U = \text{diag}(m_1, m_2, m_3)$. Using this formula, it can be shown that bimaximal mixing is equivalent to a 2-3 symmetry of the mass matrix M_ν [10], and trimaximal mixing is equivalent to a magic symmetry [11]. The 2-3 symmetry defined by the relations $(M_\nu)_{12} = (M_\nu)_{13}$ and $(M_\nu)_{22} = (M_\nu)_{33}$ is generated by a unitary matrix G_3 commuting with M_ν , and the magic symmetry characterized by the magic-square property of having equal sums for rows and columns is generated by another unitary matrix G_2 commuting with M_ν [12]. We shall refer to these symmetries as *residual symmetries* of M_ν . The residual symmetry group generated by G_2 and G_3 is a $Z_2 \times Z_2$ group. It contains the element $G_1 = G_2 G_3 = G_3 G_2$ which generates the mixing pattern of the first column of U in (1). Each G_i generates a subgroup of $Z_2 \times Z_2$ isomorphic to Z_2 .

As a matter of fact, such generators G_i can be constructed for *any* real PMNS matrix U as follows. Let v_i denote the i th column of U , then the three v_i form an orthonormal set, and the equation $U^T M_\nu U = \text{diag}(m_1, m_2, m_3)$ is equivalent to the eigenvalue equations

$M_\nu v_i = m_i v_i$. The matrix $G_i = -E + 2v_i v_i^\dagger$ has eigenvalue 1 with the eigenvector v_i , and a degenerate eigenvalue -1 for the other two eigenvectors. We may therefore choose the other two eigenvectors of G_i to be v_j and v_k , with $i \neq j \neq k \neq i$. Since G_i has the same eigenvectors as M_ν , clearly the two commute. It is also easy to verify that G_i is unitary, $G_i^2 = E$, and $G_i G_j = G_j G_i = G_k$. Hence a residual symmetry group $Z_2 \times Z_2$ generated by these G_i is present for any real PMNS matrix U . The difference between the G_i 's of different U 's is their explicit matrix form, and not the group structure. In the case of the tri-bimaximal mixing in (1), the generators can be computed to be [12]

$$G_1 = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix}, \quad G_2 = -\frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}, \quad G_3 = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (3)$$

What about residual symmetries of the charged-lepton mass matrix? Since it is diagonal and non-degenerate, any unitary matrix F that commutes with it must be diagonal with unit modulus in all its entries. We shall assume the presence of at least one residual symmetry $F \neq E$ with non-degenerate eigenvalues. This simple but powerful assumption allows us to construct a horizontal symmetry group \mathcal{G} generated by F and one or two (the third one is not independent) G_i . In other words, $\mathcal{G}^i = \{F, G_i\}$ and $\mathcal{G}^0 = \{F, G_2, G_3\}$. We shall also assume the horizontal group to be a finite group.

The reason for requiring F to be non-degenerate is to ensure $M_e M_e^\dagger$, which commutes with F , to be diagonal when F is. This is necessary since the neutrino residual symmetries G_i in (3) are defined in the basis where $M_e M_e^\dagger$ is diagonal. This is also the reason why we need to have a residual symmetry F for the charged-leptons.

The reason to take \mathcal{G} to be a finite group is economy. Since all the symmetries in \mathcal{G} other than the residual symmetries are eventually broken, there is no point to throw away more than necessary by starting with a larger group. With a finite group, especially a small one, the theory also has more predictive power. Given a F , suppose the pair $\{F, G_i\}$ generates a finite group. Among other things this means the existence of an integer m so that $(G_i F)^m = E$. Clearly this relation can no longer be satisfied if we make a small change of G_i , so the resulting new group is either infinite or very large. See Sec. 3 for further discussions on this point. Hence finite-group requirement has the power to limit G_i to a discrete number of choices.

Since \mathcal{G} is a finite group, there must be an integer n for which $F^n = E$. The corresponding group is labeled \mathcal{G}_n^α . Given an n , the three matrix elements of the diagonal F must each be an n th root of unity, hence there are $n(n-1)(n-2)$ choices of F with non-degenerate eigenvalues. In particular we need to have $n \geq 3$. Since every G_i in (3) is 2-3 symmetric, it will not produce anything new if we interchange the 2, 3 elements of F , hence there are only $n(n-1)(n-2)/2$ distinct ones. If we label them by an index a , and the corresponding diagonal matrix denoted as F_{na} , then the group generated by $\{F_{na}, G_i\}$ or $\{F_{na}, G_2, G_3\}$ may or may not be finite. If it is not finite, then we reject it and go on to another F . If it is finite, we include it in the list for \mathcal{G}_n^α .

There is the practical matter of deciding whether the group generated by $\{F, G\}$ is finite or not. A necessary condition is that GF must have a finite order m : $(GF)^m = E$. If necessary we can resort to numerical means by computing the eigenvalues of GF . For m to be finite, every one of the eigenvalues must have modulus one, and its phase angle divided by 2π must be a rational number. When computed numerically, every floating point number can be approximated by a rational number, but if the denominator of this rational number is larger than 1000, I shall declare the number to be irrational and that F rejected. Otherwise we know what m is. The resulting group, generated by $\{F, G\}$ with the relations $G^2 = F^n = (GF)^m = E$, will be denoted as $H(m, n)$ [8]. For some m and n such a group is explicitly known [9]. For others we do not even know whether the group is finite, in which case we must compute the multiplication table to find out.

For $n = 3$, there are three distinct F 's: $F_{31} = \text{diag}(1, \omega, \omega^2)$, $F_{32} = \text{diag}(\omega, 1, \omega^2)$, $F_{33} = \text{diag}(\omega^2, 1, \omega)$, where $\omega = \exp(2\pi i/3)$. The groups produced by F_{31} are $\mathcal{G}_3^1 = S_4$, $\mathcal{G}_3^2 = A_4$, $\mathcal{G}_3^3 = S_3$, and $\mathcal{G}_3^0 = S_4$. The groups produced by F_{32} and F_{33} are identical and they are $\mathcal{G}_3^1 = \mathcal{G}_3^0 = H(12, 3)$, $\mathcal{G}_3^2 = A_4$, and $\mathcal{G}_3^3 = H(6, 3)$. The group $\mathcal{G}_3^2 = A_4$ has been discussed in [13] in a similar way.

For $n = 1$ and $n = 2$, the 3×3 matrix F cannot be non-degenerate. If we ignore this requirement, then we already know the $n = 1$ result $\mathcal{G}_1^i = Z_2$ and $\mathcal{G}_1^0 = Z_2 \times Z_2$. For $n = 2$, $\mathcal{G}_2^{0,1,2}$ do not exist and $\mathcal{G}_2^3 = \{Z_2 \times Z_2, D_4\}$. These are perfectly legitimate horizontal groups but since F has degenerate eigenvalues, the mixing pattern α cannot be *automatically* produced.

It should be emphasized that although \mathcal{G}_n^α is computed when $M_e M_e^\dagger$ is diagonal, nothing really depends on it. In any other basis, the mass matrices undergo a unitary transformation,

$M \rightarrow V^\dagger M V$, but they still commute with the transformed residual symmetry operators $V^\dagger(F, G_i)V$. Moreover, the structure of \mathcal{G} remains the same with the transformed group elements.

III. DYNAMICAL MODELS

Once a horizontal group in \mathcal{G}_n^α is chosen, a dynamical model with the mixing pattern α can be produced simply by breaking all but the residue symmetries, either softly or spontaneously. Before going into the specifics there, let us first discuss how to produce a mixing pattern from *any* finite group \mathcal{G} , not necessarily one that is known to be in some \mathcal{G}_n^α .

We consider mass terms only after all right-handed fermions and heavy leptons are integrated out, in which case only the left-handed leptons $L = (e, \nu)$ and the Higgs are left. With two exceptions to be discussed below, the left-handed fermions are assigned to a three-dimensional irreducible representation (3D IR) of \mathcal{G} . For dynamical models with the presence of right-handed and/or heavy leptons, the present formalism gives only the constraint placed on the effective model after these other leptons are integrated out.

There are several requirements for \mathcal{G} to satisfy before it is qualified to be a horizontal group. Besides having a 3D IR for the left-handed fermions to occupy, it must also have an order-two element G to act as the residual symmetry of the neutrino matrix. This calls for groups with an even order. G must have one $+1$ eigenvalue with some eigenvector v which defines the partial mixing pattern, and two -1 eigenvalues.

Let us see what happens if L does not belong to a 3D IR. In that case it either contains three 1D IR or one 2D IR and a 1D IR. In the first situation if the values of G in the three 1D IR are a, b, c respectively, then G must be the diagonal matrix $G = \text{diag}(a, b, c)$. To have the correct eigenvalues, one of the three must be $+1$ and the other two -1 . The eigenvector v of eigenvalue $+1$ then has two zero entries, and the third one equals to 1. The unitary mixing matrix U containing v in one of its columns must be block-diagonal, so only two of three leptons can mix. In the second situation G must be block diagonal itself, with either $+1$ or -1 appearing in the 1×1 block. If that is $+1$ then v and the mixing matrix U are the same as in the former case. If it is -1 then v must come from the 2×2 block, hence it must have a zero entry like the third column of (1). This is why the group S_3 appears in \mathcal{G}_3^3 of (2) and nowhere else, because S_3 only has a 2D IR but not a 3D IR. Besides these two

exceptional cases, L must belong a 3D IR, as claimed.

Each of the charged-lepton mass (squared) term is of the form $C_{ija}e_i^\dagger\phi_a^e e_j$, and each of the Majorana neutrino mass term is of the form $C_{ija}\nu_i^T\phi_a^\nu\nu_j$, where $\phi_a^{e,\nu}$ are the Higgs fields, and C_{ija} are the Clebsch-Gordan coefficients. The \mathcal{G} -structure is explicit in this notation but all the Standard Model structures are understood and ignored. There may be many such mass terms, each with a different Higgs, a different Clebsch-Gordan coefficient, and a different Yukawa coupling constant. We have assumed the couplings to be linear in the Higgs; otherwise we simply consider $\phi^{e,\nu}$ to be composite fields. The representation of the Higgs can be either reducible or irreducible, as long as each of the mass terms is invariant under \mathcal{G} .

Under a transformation induced by $g \in \mathcal{G}$, $\nu \rightarrow g'\nu$, $e \rightarrow g'e$, $\phi_a \rightarrow g''\phi_a$, where g' is a 3D IR of g , and g'' is the appropriate representation for the \mathcal{G} -multiplet ϕ , which can be either reducible or irreducible. The invariance of the mass terms demands $C_{ija} = g'_{ik}g'_{jl}g''_{ab}C_{klb}$ for the neutrinos and $C_{ija} = g'^{*}_{ik}g'_{jl}g''_{ab}C_{klb}$ for the charged-leptons.

The introduction of vacuum expectation values $\langle\phi\rangle$ breaks \mathcal{G} down to the residue symmetries. The residual symmetry of neutrinos is by construction G , but we must still decide on the residue symmetry F . In principle, it can be any element of \mathcal{G} as long F' has three distinct eigenvalues. If its order is n , then we must have $n \geq 3$ for F to have all distinct eigenvalues.

The mass matrices $(M_e M_e^\dagger)_{ij}$ and $(M_\nu)_{ij}$ are of the form $C_{ija}\langle\phi_a\rangle$. A sufficient condition for these mass terms to be invariant under the residue symmetries F and G is $F''\langle\phi^e\rangle = \langle\phi^e\rangle$ and $G''\langle\phi^\nu\rangle = \langle\phi^\nu\rangle$. Now go to the basis where F' is diagonal. Since its eigenvalues are all distinct and it commutes with $M_e M_e^\dagger$, the charged-lepton mass matrix must also be diagonal in that basis. The eigenvector v of G' with eigenvalue +1 then defines a partial mixing pattern, as it occupies one column of the mixing matrix U .

These arguments apply just as well to the special case when $\mathcal{G} = \mathcal{G}_n^\alpha$, except that in this case we simply take $F' = F_{na}$ and $G' = G_i$ when $\alpha = i$. When $\alpha = 0$, we will repeat the construction for two different i 's.

If $\langle\phi^e\rangle$ and/or $\langle\phi^\nu\rangle$ is a \mathcal{G} -triplet in the same representation as the leptons, or tensors of such triplets, then much more can be said because we know then $F'' = F' = F_{na}$ and/or $G'' = G' = G_i$. Since the diagonal matrix F_{na} has three different entries along its diagonal, this invariance for $\langle\phi^e\rangle$ is possible only when one of the diagonal entries of F_{na} is 1, and

the components of the $\langle\phi^e\rangle$ facing the other two entries are zero. Thus dynamical models constructed from a horizontal group without an entry 1 in its F_{na} to start with may not have a triplet Higgs of this kind with non-zero expectation values. On the neutrino side, since $G_i v_i = v_i$, $\langle\phi\rangle$ must be proportional to this v_i . This automatically prevents a full symmetry pattern $\alpha = 0$ to be obtained if triplet Higgs are present because $\langle\phi\rangle$ cannot be proportional to two different v_i 's.

In the remainder of this section, two examples of $\mathcal{G} = A_4$ from the literature will be taken to illustrate some of these points.

We see from (2) that A_4 comes from \mathcal{G}_3^2 , hence the only residual symmetry we can expect to obtain from this group without tuning parameters is the magic symmetry corresponding to trimaximal mixing with $i = 2$. Let us now look at the specific example taken from Ref. [14], where the Higgs for charged leptons (called Φ) is a (\mathcal{G} -)triplet with expectation values $\langle\Phi\rangle = v(1, 1, 1)^T$, and M_e is not diagonal. Note that this is the Higgs coupling the left-handed charged fermions to the right-handed ones, so it can be related to our Higgs ϕ^e only after the right-handed leptons have been integrated out. The resulting $\langle\phi^e\rangle$ is the tensor product of two triplets. On the neutrino side, there is a triplet Higgs field called $\xi_{4,5,6}$ and three $\underline{1}$, $\underline{1}'$, $\underline{1}''$ Higgs fields called ξ_1, ξ_2, ξ_3 , with expectation values $(\langle\xi_4^0\rangle, 0, 0)^T$ and $\langle\xi_1^0\rangle, \langle\xi_2^0\rangle, \langle\xi_3^0\rangle$ respectively.

To compare with the general theory discussed above we must first transform everything to a basis where $M_e M_e^\dagger$ is diagonal. The unitary matrix to do that is

$$V = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}. \quad (4)$$

The transformed expectation values, distinguished by a hat, are $\langle\hat{\Phi}\rangle = \sqrt{3}v(1, 0, 0)^T$ and $\langle\hat{\xi}_{4,5,6}\rangle = (\langle\xi_4^0\rangle/\sqrt{3})(1, 1, 1)^T$. Thus $\langle\hat{\Phi}\rangle$ is indeed invariant under $F'' = F_{31} = \text{diag}(1, \omega, \omega^2)$, and $\langle\hat{\xi}_{4,5,6}\rangle$ is indeed invariant under $G'' = G_2$ with eigenvalue 1, as the general theory indicates. As for the three singlet representations, we have to know the values of G'' . This is given in Refs. [13] and [15] and they are all 1 for $\underline{1}, \underline{1}'$, and $\underline{1}''$. Hence $\langle\hat{\xi}_{1,2,3}\rangle$ are invariant under G'' as well, as the general theory demands. With all these fulfilled, the mixing pattern should correspond to a trimaximal mixing ($i = 2$), with a magic neutrino mass matrix. This can be seen to be the case in CEQ. (19) of Ref. [14].

The second example is taken from Ref. [16]. In this case M_e is diagonal so the general

theory should apply without a transformation. On the charged lepton side, there are three Higgs belonging to $\underline{1}$, $\underline{1}'$, $\underline{1}''$ respectively, whose expectation values are $\langle\phi^e\rangle = v_1, v_2, v_3$. Since $F''[\underline{1}, \underline{1}', \underline{1}''] = [1, \omega^2, \omega]$ [13, 15], $\langle\phi^e\rangle$ is not invariant under F'' but $\langle\phi^e\rangle\langle\phi^e\rangle^*$ is, which is all that counts for $M_e M_e^\dagger$. On the neutrino side, a type-I seesaw mechanism is invoked. The Higgs which couples the light to the heavy neutrinos belongs to a $\underline{3}$, with expectation values $h \equiv (u_1, u_2, u_3)^T$. After integrating out the heavy neutrino, the effective Higgs expectation values $\langle\phi^\nu\rangle$ used to calculate M_ν is $h \otimes h$. h is invariant under $G'' = G_2$ if $u_1 = u_2 = u_3$. In that case G_2 is a residual symmetry of M_ν and the neutrino mass matrix should be magic. This can be seen to be true from eq. (5) of Ref. [16]. On the other hand, if $u_1 = u_2 = u \neq u_3$ is assumed, then h is not invariant under G_2 , so G_2 is no longer a residual symmetry, and M_ν is not expected to be magic. This can also be verified from eq. (8) of Ref. [16].

IV. CONCLUSION

We have studied the symmetry of neutrino mixing, assuming one residual symmetry each survives for the charged leptons and the neutrinos after the spontaneous breakdown from a finite horizontal symmetry group. The problem is studied both from the bottom up and from the top down. In the former approach, starting from any one of the three partial mixing patterns defined by (1), a list of horizontal groups labeled by n can be constructed to yield a specific pattern, and the result for $n = 3$ is explicitly shown in eq. (3). In the latter approach, a necessary condition for a finite group to be horizontal is given, and a general recipe is provided to decide what mixing patterns it contains, and how to produce an effective dynamical models giving rise to such a pattern automatically after the right-handed and the heavy leptons are integrated out.

It is natural to ask whether this approach also works for quark mixing. Since the Cabibbo angle is small, the CKM matrix is not as neat as (1). In any case, it is given numerically and no analytical approximation such as (1) is known, so the finite horizontal groups for it are much harder to find. At this point it is not clear whether such a finite group exists or not for the mixing of three quarks. If it does, then the discreteness of the allowed mixing patterns for any horizontal group promises the exciting possibility of determining three CKM mixing parameters once the fourth of them is used to fix which discrete set it belongs to. To test whether a finite horizontal group exists for quark mixing, I consider the much simpler

situation of mixing only two quarks, in which case the mixing matrix is controlled by a single parameter λ , the sine of the Cabibbo angle. Details will appear elsewhere. I find that this mixing matrix can indeed be accommodated by the dihedral group D_m , for a discrete set of Cabibbo angles given by the formula $\theta_c = \pi/2m$. For $m = 7$, the value of λ is $\lambda = 0.2225$, which is to be compared with the Particle Data Group value of either $\lambda = 0.2272 \pm 0.0010$, or $\lambda = 0.2262 \pm 0.0014$, from two different fits. Although the predicted value is not close enough to the experimental result, it is nevertheless way within $O(\lambda^2)$ of it, an error which we might expect to make by ignoring the third quark in the mixing. This result gives some hope for the feasibility of the scheme, but to be sure we need to find at least one finite group which contains the mixing of three quarks. This investigation is underway.

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